On the equality of the PSD and the LS-HE T-test statistics

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Here we show that the T-test statistics of the LS-HE can include the PSD (of Fourier Transform) as a special case. The T-test statistic is obtained using the following formula

$$\mathrm{T} = \hat{e}_0^T Q_y^{-1} C \left(C^T Q_y^{-1} P_A^{\perp} C\right)^{-1} C^T Q_y^{-1} \hat{e}_0$$

where y is the vector of time series observations, $\hat{e}_0 = P_A^{\perp} y$ is the least squares residuals under the null hypothesis H_0, $P_A^{\perp}$ is an orthogonal projector, Q_y is the covariance matrix of observations, and C is a matrix containing two columns of sin and cos, as the signal to be tested. This test statistic, having chi-square distribution, can be tested in a given confidence level (2 is the columns of C)

$$\mathrm{T} \sim \chi^2(2,0)$$

A <u>special case</u> of the above formula can deal with a zero-mean time series. For this situation, the design matrix A is not present, so $A = 0$ and therefore $P_A^{\perp} = I$ is an identity matrix. This will then give $\hat{e}_0 = y$. Further simplification regards the situation where the time series consist of unit white noise resulting in $Q_y = I$. The T-test statistics become then

$$\mathrm{T} = y^T C (C^T C)^{-1} C^T y$$

This can, in fact, be shown to be identical to the multiplication of PSD by a factor of 2, i.e. as

$$\mathrm{T}(k) = 2 S_{yy}(k) = \frac{2}{m\Delta t} |Y(k)\,|^2$$

Proof. To be consistent with the PSD of the week 'Signal Processing', we consider the time series as $y = [y_0, y_1, \dots, y_{m-1}]$ at time instances $y = [t_0, t_1, \dots, t_{m-1}] = [0,1, \dots, m-1]$. We then have $T = m\Delta t = m$. The matrix C is then as follows:

$$C = \begin{bmatrix} \cos\omega t_0 & \sin\omega t_0 \\ \cos\omega t_1 & \sin\omega t_1 \\ \vdots & \vdots \\ \cos\omega t_{m-1} & \sin\omega t_{m-1} \end{bmatrix} = \begin{bmatrix} \cos 0\omega & \sin 0\omega \\ \cos\omega & \sin\omega \\ \vdots & \vdots \\ \cos(m-1)\omega & \sin(m-1)\,\omega \end{bmatrix}$$

which gives

$$C^T C = \begin{bmatrix} \sum_{i=0}^{m-1} \cos^2(i\omega) & \sum_{i=0}^{m-1} \sin(i\omega)\cos(i\omega) \\ \sum_{i=0}^{m-1} \sin(i\omega)\cos(i\omega) & \sum_{i=0}^{m-1} \sin^2(i\omega) \end{bmatrix} = \begin{bmatrix} cc & cs \\ cs & ss \end{bmatrix}$$

We then define the following frequencies as we have in Fourier Transform (note that LS-HE does not this requirement, so in principle any frequency can be tested):

$$\omega_k = 2\pi f_k = 2\pi\frac{k}{T} = \frac{2\pi k}{m}, k = 0,1, \dots, m-1$$

We therefore have

$$cs(\omega_k) = \sum_{i=0}^{m-1} \sin\left(\frac{2\pi i k}{m}\right)\cos\left(\frac{2\pi i k}{m}\right)$$

Using the identity

$$\sin(a)\cos(a) = \frac{1}{2}\sin(2a)$$

it follows that

$$cs(\omega_k) = \frac{1}{2}\sum_{i=0}^{m-1} \sin(\frac{4\pi ik}{m}) = \frac{1}{2}\sum_{i=0}^{m-1} \sin(2\pi\frac{2ik}{m})$$

or (let us assume m is an even number, when m is odd a similar procedure can be repeated)

$$cs(\omega_k) = \frac{1}{2}\sum_{i=0}^{\frac{m}{2}-1} \sin(2\pi\frac{2ik}{m}) + \frac{1}{2}\sum_{i=\frac{m}{2}}^{m-1} \sin(2\pi\frac{2ik}{m})$$

or

$$cs(\omega_k) = \frac{1}{2}\sum_{i=0}^{\frac{m}{2}-1} \sin(2\pi\frac{2ik}{m}) + \frac{1}{2}\sum_{i=0}^{\frac{m}{2}-1} \sin(2\pi\frac{2(i+\frac{m}{2})k}{m})$$

or

$$cs(\omega_k) = \frac{1}{2}\sum_{i=0}^{\frac{m}{2}-1} \sin(2\pi\frac{2ik}{m}) + \frac{1}{2}\sum_{i=0}^{\frac{m}{2}-1} \sin(2\pi\frac{2ik+mk}{m})$$

or

$$cs(\omega_k) = \frac{1}{2}\sum_{i=0}^{\frac{m}{2}-1} \sin(2\pi\frac{2ik}{m}) + \frac{1}{2}\sum_{i=0}^{\frac{m}{2}-1} \sin(2\pi k + 2\pi\frac{2ik}{m})$$

which gives

$$cs(\omega_k) = \sum_{i=0}^{\frac{m}{2}-1} \sin(2\pi\frac{2ik}{m})$$

We know that $\sin(2\pi\frac{2ik}{m})$ at $i = 0$ is zero and therefore

$$cs(\omega_k) = \sum_{i=1}^{\frac{m}{2}-1} \sin(2\pi\frac{2ik}{m})$$

Writing out the series as (put some of the terms together: the first and last terms, the second and the last but one, …), we will have

$$cs(\omega_k) = \sin(2\pi\frac{2k}{m}) + \sin(2\pi\frac{2(\frac{m}{2}-1)k}{m}) + \sin(2\pi\frac{4k}{m}) + \sin(2\pi\frac{2(\frac{m}{2}-2)k}{m}) + \cdots$$

or

$$cs(\omega_k) = \sin(2\pi\frac{2k}{m}) + \sin(2k\pi - 2\pi\frac{2k}{m}) + \sin(2\pi\frac{4k}{m}) + \sin(2k\pi - 2\pi\frac{4k}{m}) + \cdots$$

Using the identity $\sin(2k\pi - \alpha) = -\sin(\alpha)$ we have

$$cs(\omega_k) = \sin(2\pi\frac{2k}{m}) - \sin(2\pi\frac{2k}{m}) + \sin(2\pi\frac{4k}{m}) - \sin(2\pi\frac{4k}{m}) + \cdots = 0 + 0 + \cdots = 0$$

In a similar manner we can prove that

$$cc(\omega_k) = ss(\omega_k) = \frac{m}{2}$$

which gives

$$\begin{bmatrix} cc & cs \\ cs & ss \end{bmatrix} = \frac{m}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} cc & cs \\ cs & ss \end{bmatrix}^{-1} = \frac{2}{m}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can also simply show that

$$C^T y = \begin{bmatrix} \sum_{i=0}^{m-1} y_i \cos(\frac{2\pi ik}{m}) \\ \sum_{i=0}^{m-1} y_i \sin(\frac{2\pi ik}{m}) \end{bmatrix}$$

which gives

$$T(\omega_k) = y^T C (C^T C)^{-1} C^T y$$

and finally, it results in

$$T(\omega_k) = T(k) = \frac{2}{m}\left(\left(\sum_{i=0}^{m-1} y_i \cos(\frac{2\pi ik}{m})\right)^2 + \left(\sum_{i=0}^{m-1} y_i \sin(\frac{2\pi ik}{m})\right)^2\right)$$

The Power Spectral Density (PSD) is obtained as

$$S_{yy}(k) = \frac{1}{m\Delta t}|Y(k)\ |^2 = \frac{1}{m}|Y(k)\ |^2$$

where $Y(k)$ is the Fourier Transform of y

$$Y(k) = Y_k = \Delta t \sum_{i=0}^{m-1} y_i e^{-j\frac{2\pi ki}{m}} = \sum_{i=0}^{m-1} y_i e^{-j\frac{2\pi ki}{m}}$$

Using the Euler formula $e^{j\theta} = \cos\theta + j\sin\theta$, we have

$$Y(k) = Y_k = \sum_{i=0}^{m-1}\left(y_i \cos\left(\frac{2\pi ki}{m}\right) - j\, y_i \sin(\frac{2\pi ki)}{m})\right)$$

or

$$Y(k) = Y_k = \sum_{i=1}^{m-1} y_i \cos\left(\frac{2\pi ki}{m}\right) - j\sum_{i=0}^{m-1}\ y_i \sin(\frac{2\pi ki}{m})$$

This will then give

$$S_{yy}(k) = \frac{1}{m}|Y(k)\ |^2 = \frac{1}{m}\left(\sum_{t=1}^{m} y_i \cos\left(\frac{2\pi ki}{m}\right)\right)^2 + \frac{1}{m}\left(\sum_{t=1}^{m} y_i \sin\left(\frac{2\pi ki}{m}\right)\right)^2$$

When compared with

$$T(k) = \frac{2}{m}\left(\left(\sum_{i=0}^{m-1} y_i \cos(\frac{2\pi ik}{m})\right)^2 + \left(\sum_{i=0}^{m-1} y_i \sin(\frac{2\pi ik}{m})\right)^2\right)$$

It follows that

$$\boxed{T(k) = 2S_{yy}(k)}$$